Closed Form Solution to Natural Image Matting

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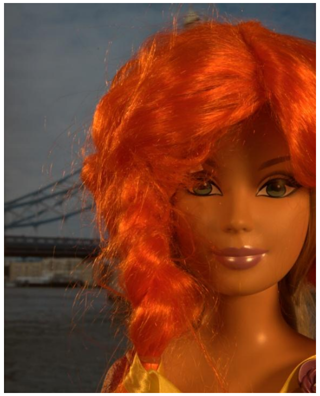
What Is Image Matting?

A **composite** image is treated as a linear combination of **foreground** and **background** images

Compositing equation: $I_i = \alpha_i F_i + (1 - \alpha_i) B_i$

- I_i : intensity of composite pixel
- F_i : intensity of foreground pixel
- B_i : intensity of background pixel
- $\alpha_i \in [0,1]$: alpha matte of pixel



Query *composite* image

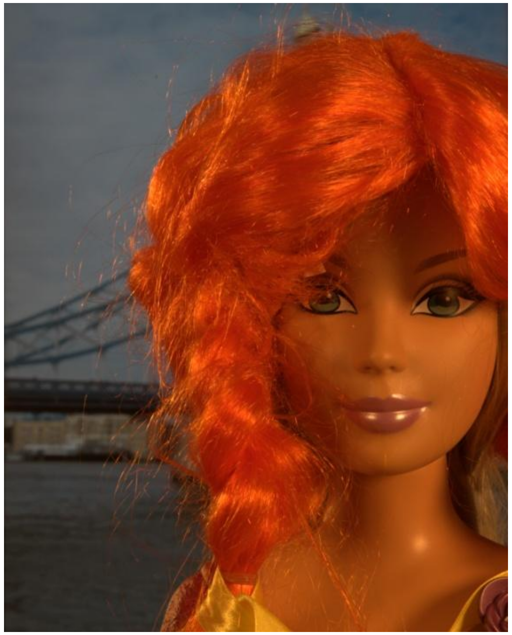
Ground truth alpha matte

The **alpha matte** of a pixel defines the contributions from foreground/background

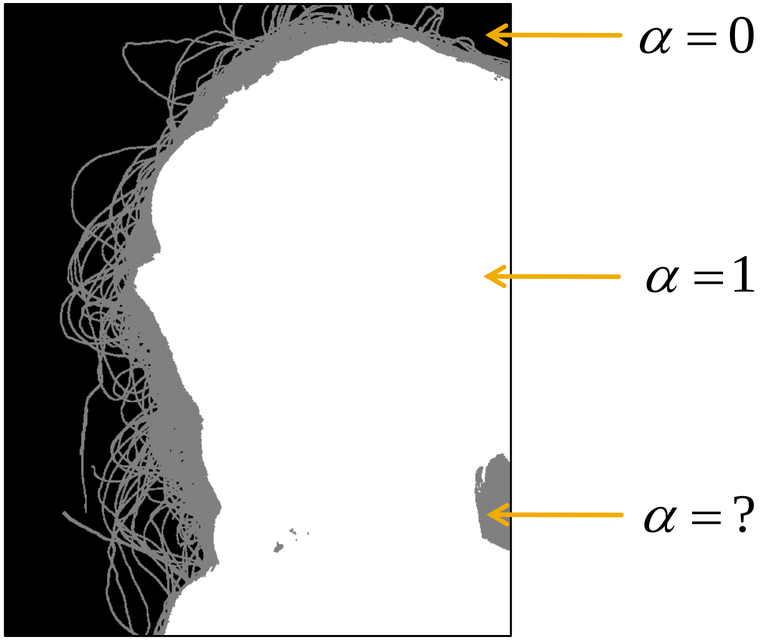
User Interaction for Matting

$$I_i = \alpha_i F_i + (1 - \alpha_i) B_i$$ → 1 equation in 3 unknowns (grayscale images)
3 equations in 7 unknowns (color images)

The image matting problem is severely under-constrained!



Query *composite* image



Trimap

User interaction helps embed constraints into the image

Comparison with other approaches

Bayes Matting:

- Bayesian matting uses GMM for foreground background.
- Generative model used to estimate F_i, B_i and α_i.
- No smoothness priors used.
- Each parameter estimated iteratively using non-linear optimization.
- Works well when color distributions do not overlap.
- Requires small unknown region in trimap.

Poisson Matting:

- Solves Poisson equation with matte gradient field and boundary conditions.
- When F_i, B_i, not sufficiently smooth inside unknown region, matte incorrect.
- Such cases requires heavy interaction (local Poisson matting).

Contributions

- Closed form solution for extracting alpha matte.
- Elimination of F_i, B_i in optimization resulting in quadratic expression in α_i.
- Modest user input sufficient through guided spectral analysis.

Case 1: Gray scale

- Assume *F* and *B* approximately constant over small window, *w*.

$$\alpha_i = aI_i + b, \forall i \in w. \quad a = \frac{1}{F-B}, b = \frac{-B}{F-B}.$$

Knowns: 9, Unknowns:~~27~~ 11

Want to eliminate *a*, *b*.

- So, finding matte amounts to minimizing:

$$J(\alpha, a, b) = \sum_{j \in I} \left(\sum_{i \in w_j} (\alpha_i - a_j I_i - b_j)^2 + \varepsilon a_j^2 \right) = \sum_{j \in I} \left\| G_j \begin{bmatrix} a_j \\ b_j \end{bmatrix} - \bar{\alpha}_j \right\|_2^2$$

- Regularization for numerical stability, and to avoid over-fitting.
- Window size typically 3x3.
- Overlap of windows enables propagation of information between neighbors.

Case 1: Gray scale

Theorem 1: *Define* $J(\alpha) = \min_{a,b} J(\alpha, a, b)$. *Then,*

$$J(\alpha) = \alpha^T L \alpha,$$

where L *is an* $N \times N$ *matrix, whose* (i, j)*-th entry is:*

$$\sum_{k|(i,j)\in w_k} \left(\delta_{ij} - \frac{1}{|w_k|} \left(1 + \frac{1}{\frac{\varepsilon}{|w_k|} + \sigma_k^2} (I_i - \mu_k)(I_j - \mu_k) \right) \right)$$

Proof:

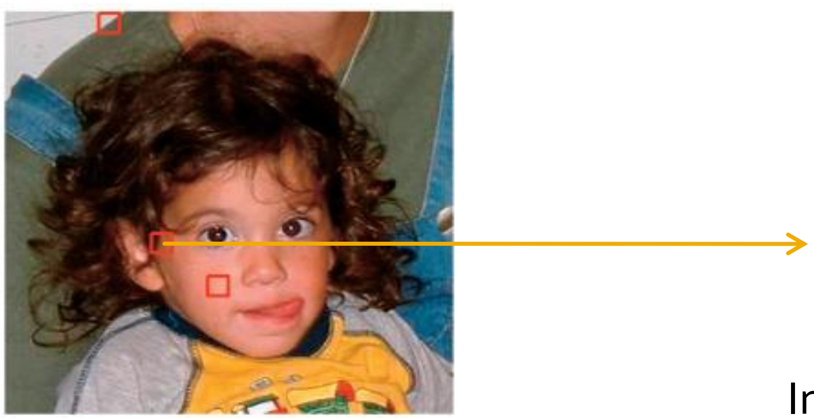
$$\min_{a,b} J(\alpha, a, b) = \min_{a,b} \sum_{j \in I} \left\| G_j \begin{bmatrix} a_j \\ b_j \end{bmatrix} - \overline{\alpha}_j \right\|_2^2 = \sum_{j \in I} \overline{\alpha}_j^T P_j^T P_j \overline{\alpha}_j = \alpha^T L \alpha$$

Where, $$P_j = I - G_j (G_j{}^T G_j)^{-1} G_j{}^T.$$

Case 2: Color Image

Color compositing equation: $I_i^c = \alpha_i F_i^c + (1-\alpha_i) B_i^c$

The intensities of each image layer can be assumed to be **locally linear** in RGB space



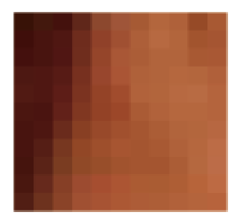
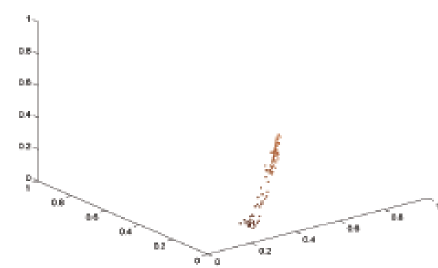


Image patch W_i



Distribution of RGB values for pixels in W_i

The intensities of **each** image layer are assumed to obey **color line models**

$$F_i^c = \beta_i^F F_1^c + (1-\beta_i^F) F_2^c$$

$$B_i^c = \beta_i^B B_1^c + (1-\beta_j^B) B_2^c$$

Case 2: Color Image

Theorem 2 *If the foreground and background colors in a window satisfy the color line model we can express*

$$\alpha_i = \sum_c a I_i^{\,c} + b, \forall i \in w.$$

Proof:

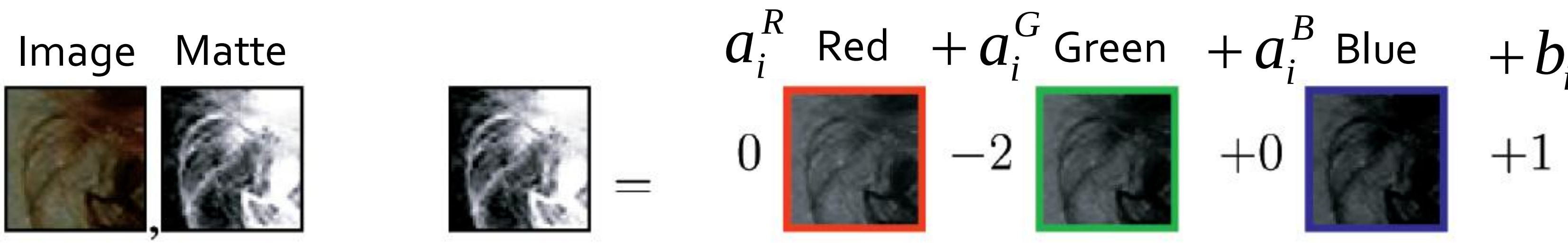
$$I_i^{\,c} = \alpha_i (F_2^{\,c} - B_2^{\,c}) + \alpha_i \beta_i^F (F_1^c - F_2^{\,c}) + (1-\alpha_i)\beta_i^B (B_1^c - B_2^{\,c}) + B_2^{\,c}$$

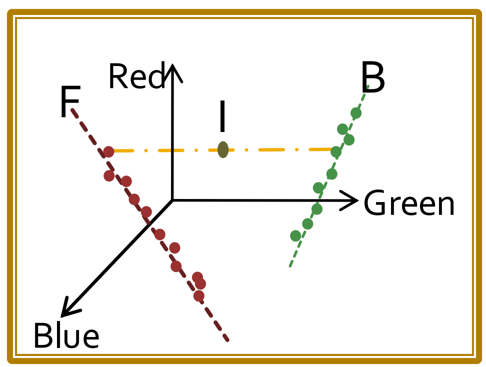
$$\Rightarrow \begin{bmatrix} \alpha_i \\ \alpha_i \beta_i^F \\ \alpha_i (1-\beta_i^B) \end{bmatrix} = \begin{bmatrix} F_2^{\,r} - B_2^{\,r} & F_1^{\,r} - F_2^{\,r} & B_1^{\,r} - B_2^{\,r} \\ F_2^{\,g} - B_2^{\,g} & F^g{}_1 - F_2^{\,g} & B_1^{\,g} - B_2^{\,g} \\ F_2^{\,b} - B_2^{\,b} & F_1^{\,b} - F_2^{\,b} & B_1^{\,b} - B_2^{\,b} \end{bmatrix}^{-1} \begin{bmatrix} I_i^r - B_2^r \\ I_i^g - B_2^g \\ I_i^b - B_2^b \end{bmatrix}$$

$$\alpha_i = a^r I_i^r + a^g I_i^g + a^b I_i^b + b$$

The alpha matte of a pixel is an **affine function** of the pixel’s image intensities

Affine Functions for Alpha Matting





Affine functions can be used to describe the alpha mattes for **complex patches** too

Closed Form Solution to Matting

Regularized cost function for image matting:

$$J_\varepsilon(\alpha, a, b) = \sum_i \left[\sum_{j \in W_i} \left(\alpha_j - a_i^r I_j^r - a_i^g I_j^g - a_i^b I_j^b - b_i \right)^2 + \varepsilon \left(a_i^r + a_i^g + a_i^g \right)^2 \right]$$

Eliminate the color line models from the cost function

$$(a_i^r, a_i^g, a_i^b, b_i) = \underset{(a^r, a^g, a^b, b)}{\operatorname{argmin}} \left[\sum_{j \in W_i} \left(\alpha_j - a^r I_j^r - a^g I_j^g - a^b I_j^b - b \right)^2 + \varepsilon \left(a^r + a^g + a^b \right) \right] = P \begin{bmatrix} \alpha_{j_1} \\ \vdots \\ \alpha_{j_{|W_i|}} \end{bmatrix}$$

Substitute this into the original cost function to get a quadratic cost function

$$J_\varepsilon(\alpha) = \alpha^T \underbrace{L}\alpha$$

Matting Laplacian

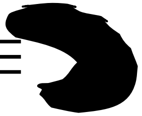
The alpha mattes are estimated as the solution of a linear system of equations

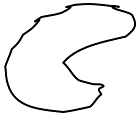
Constraints and UI

Input:

Image+ user scribbles

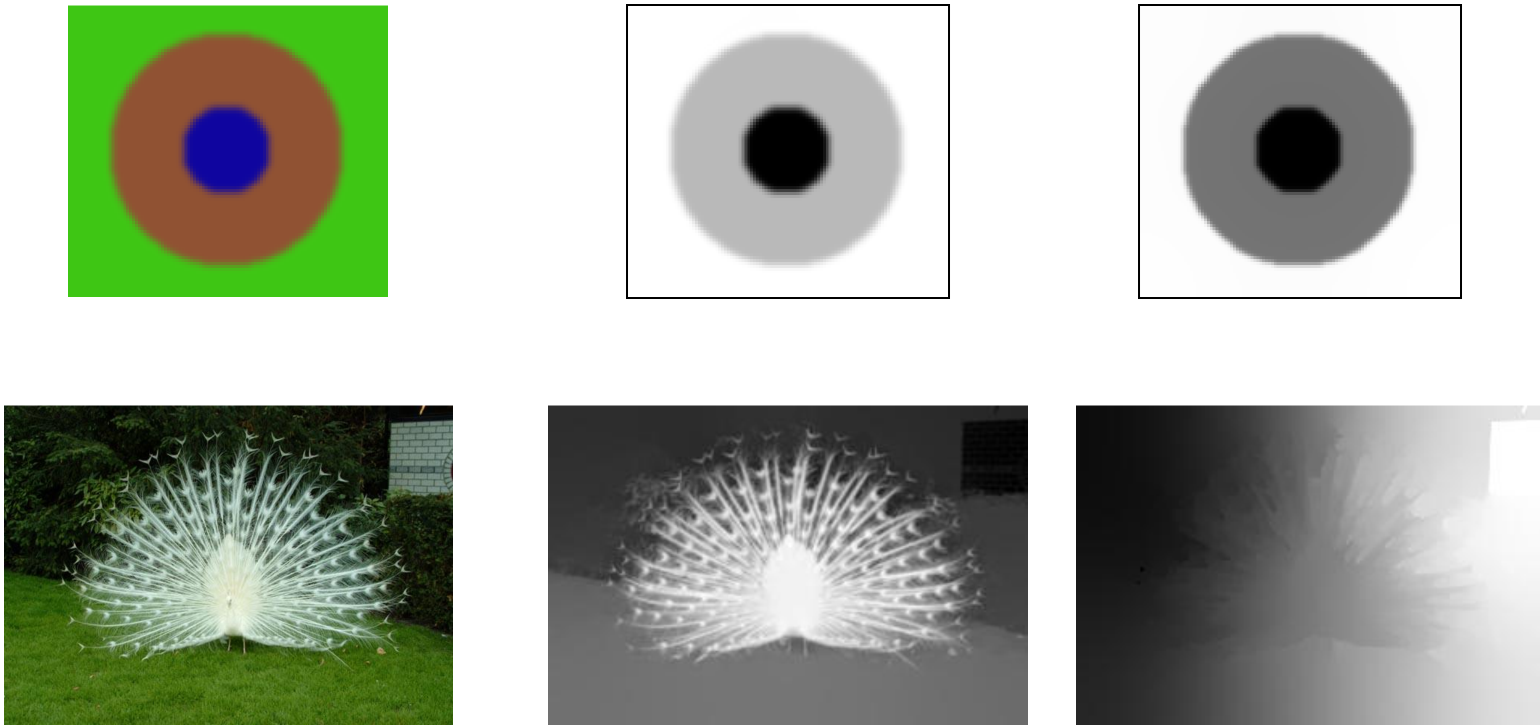
$$\alpha = \arg\min \alpha^T L \alpha$$

$$s.t. \quad \alpha_i = 0, \quad i \in$$ 

$$\alpha_i = 1, \quad i \in$$ 

Where, $L(i,j) \propto \sum_{k|(i,j)\in w_k} -\left(1 + (I_i - \mu_k)^T (\Sigma_k + \mathrm{I}_3)^{-1} (I_j - \mu_k)\right)$

Spectral Analysis

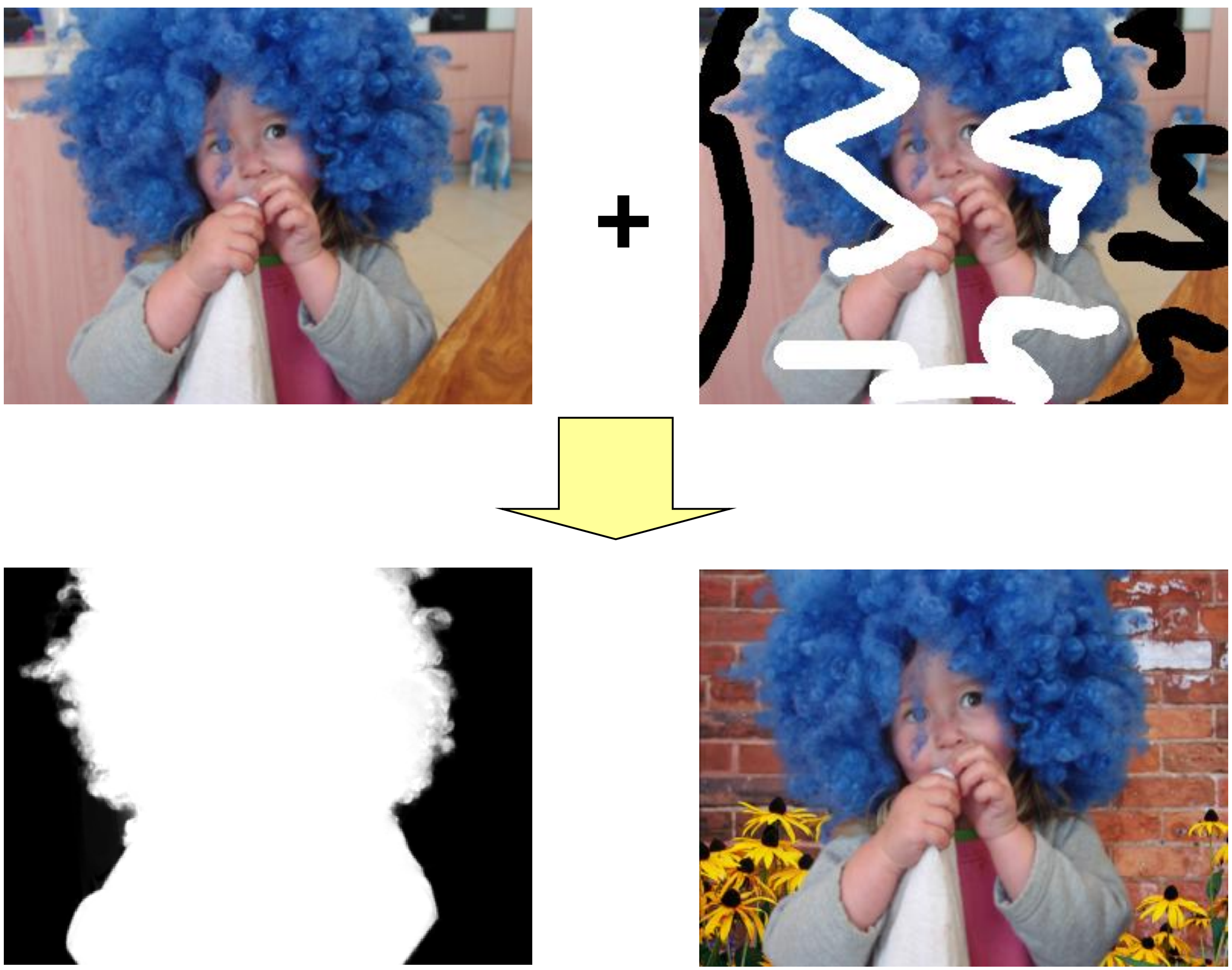


Input image

Matting Eigenvectors

Global- σ Eigenvectors

Results



Conclusions

- Analytically eliminate F,B and obtain quadratic cost.
- Solve efficiently using linear algebra.
- Results are provably correct.
- Connection to spectral segmentation.

Questions?