Style-Based Inverse Kinematics

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CS-184: Computer Graphics

Lecture 20: Forward and Inverse Kinematics

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Slides based on those of James O'Brien

Announcements

Final Project: multiple due dates

- Project proposal due Wed Nov 17, 11pm
- Progress report 1 due Mon Nov 22, 11pm
- Progress report 2 due Wed Dec 1, 11pm
- Final report due Wed Dec 8, 11pm

Today

Forward kinematics

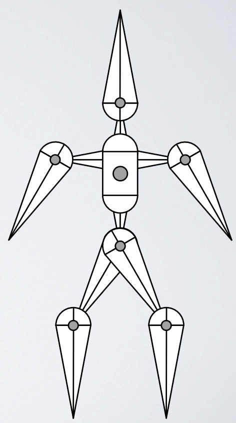
Inverse kinematics

- Pin joints
- Ball joints
- Prismatic joints

Forward Kinematics

Articulated skeleton

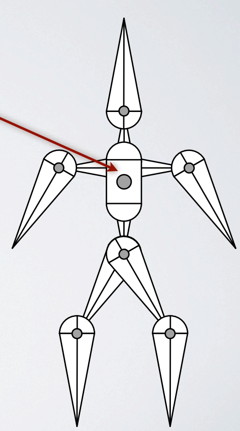
- Topology (what's connected to what)
- Geometric relations from joints
- Independent of display geometry
- Tree structure
  - Loop joints break "tree-ness"



Forward Kinematics

Root body

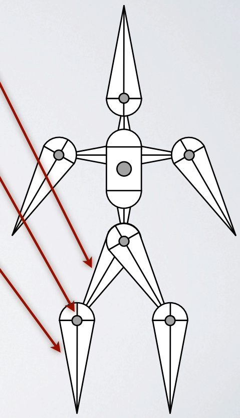
- Position set by "global" transformation
- Root joint
  - Position
  - Rotation
- Other bodies relative to root
- ***Inboard*** **toward the root**
- ***Outboard*** **away from root**



Forward Kinematics

A joint

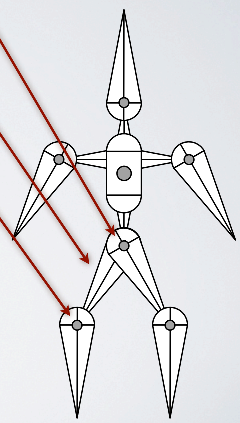
- Joint's inboard body
- Joint's outboard body



Forward Kinematics

A body

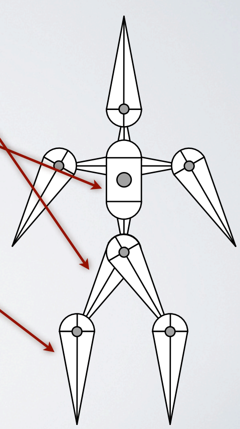
- Body's inboard joint
- Body's outboard joint
  - May have several outboard joints



Forward Kinematics

A body

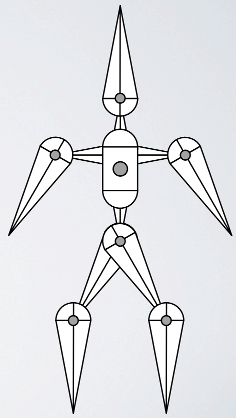
- Body's inboard joint
- Body's outboard joint
  - May have several outboard joints
- Body's parent
- Body's child
  - May have several children



Forward Kinematics

Interior joints

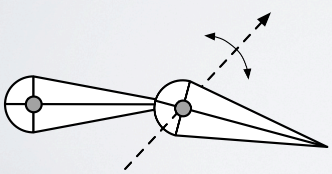
- Typically not 6 DOF joints
- Pin - rotate about one axis
- Ball - arbitrary rotation
- Prism - translation along one axis

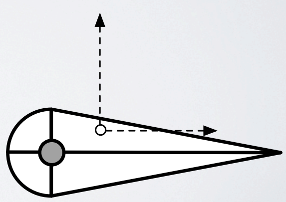


Forward Kinematics

Pin Joints

- Translate inboard joint to local origin
- Apply rotation about axis
- Translate origin to location of joint on outboard body

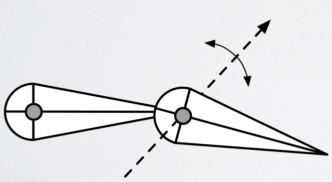


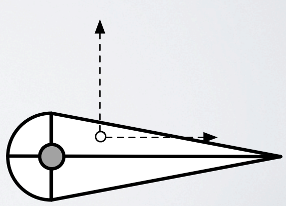


Forward Kinematics

Ball Joints

- Translate inboard joint to local origin
- Apply rotation about arbitrary axis
- Translate origin to location of joint on outboard body

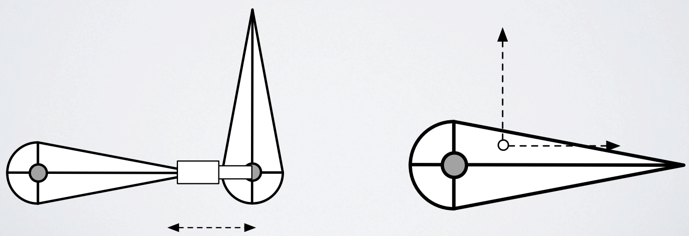




Forward Kinematics

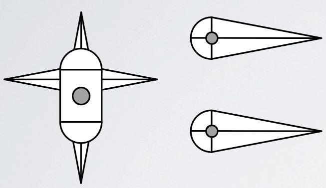
Prismatic Joints

- Translate inboard joint to local origin
- Translate along axis
- Translate origin to location of joint on outboard body



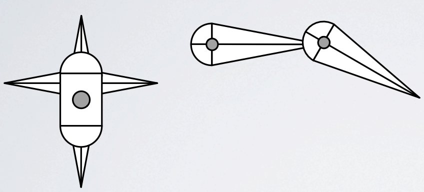
Forward Kinematics

Composite transformations up the hierarchy



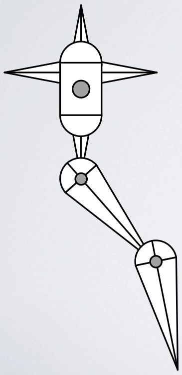
Forward Kinematics

Composite transformations up the hierarchy



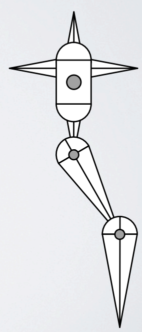
Forward Kinematics

Composite transformations up the hierarchy



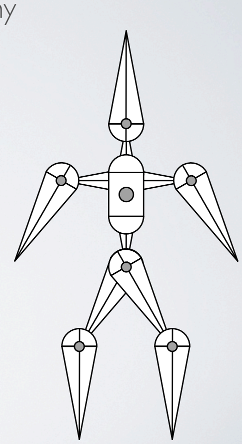
Forward Kinematics

Composite transformations up the hierarchy



Forward Kinematics

Composite transformations up the hierarchy



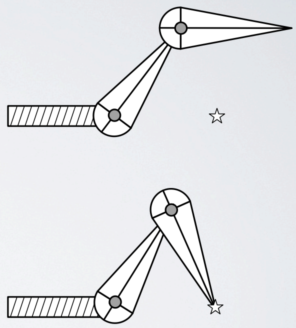
Inverse Kinematics

Given

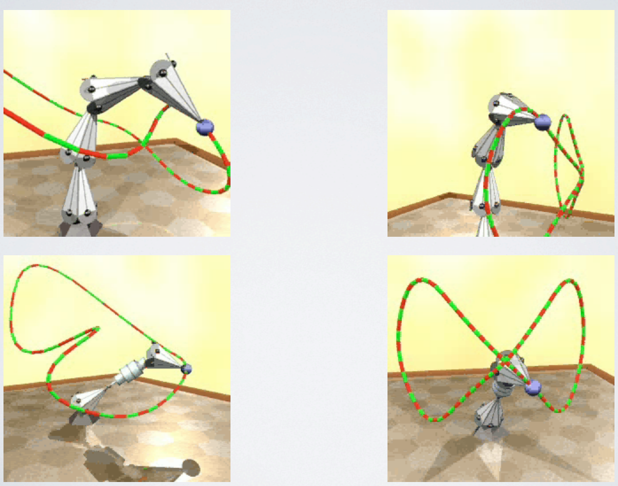
- Root transformation
- Initial configuration
- Desired end point location

Find

- Interior parameter settings



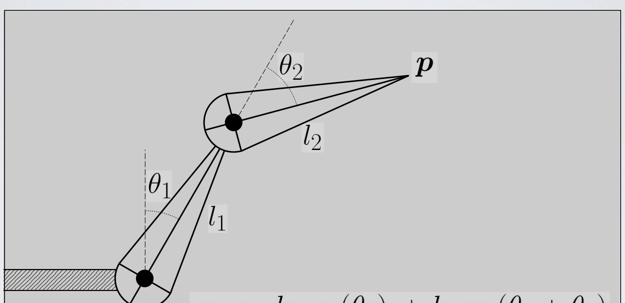
Inverse Kinematics



Egon Pasztor

Inverse Kinematics

A simple two segment arm in 2D



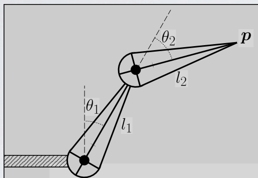
$$p_z = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$p_x = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

Warning: Z-up Coordinate System

Inverse Kinematics

Direct IK: solve for the parameters



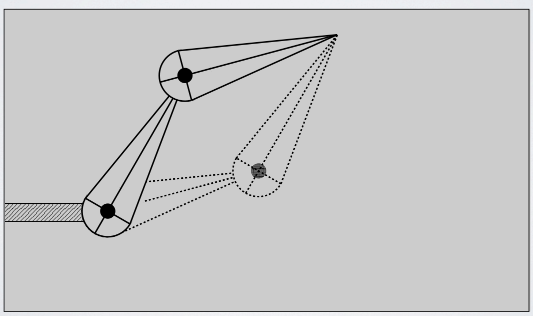
$$\theta_2 = \cos^{-1}\left(\frac{p_z^2 + p_x^2 - l_1^2 - l_2^2}{2l_1l_2}\right)$$

$$\theta_1 = \frac{-p_z l_2 \sin(\theta_2) + p_x(l_1 + l_2\cos(\theta_2))}{p_x l_2 \sin(\theta_2) + p_z(l_1 + l_2\cos(\theta_2))}$$

Inverse Kinematics

Why is the problem hard?

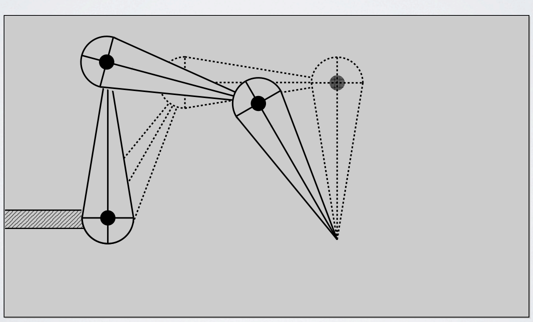
- Multiple solutions separated in configuration space



Inverse Kinematics

Why is the problem hard?

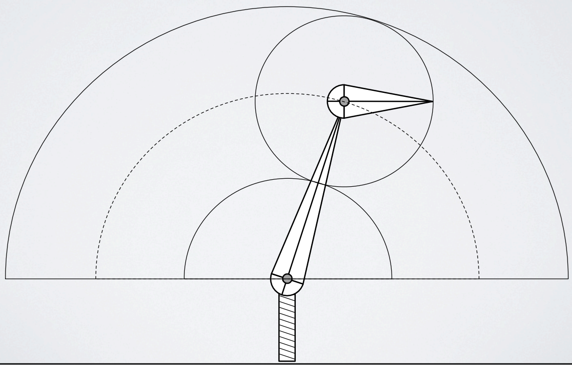
- Multiple solutions connected in configuration space



Inverse Kinematics

Why is the problem hard?

- Solutions may not always exist



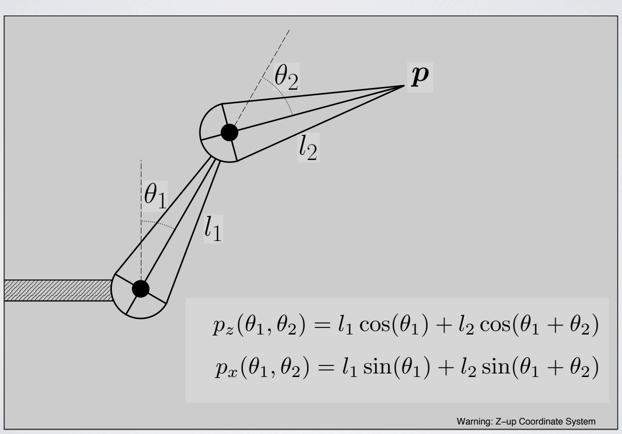
Inverse Kinematics

Numerical Solution

- Start in some initial configuration
- Define an error metric (**e.g.** goal pos - current pos)
- Compute Jacobian of error w.r.t. inputs
- Apply Newton's method (or other procedure)
- Iterate...

Inverse Kinematics

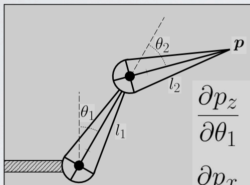
Recall simple two segment arm:



$$p_z(\theta_1, \theta_2) = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$
$$p_x(\theta_1, \theta_2) = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

Inverse Kinematics

We can write of the derivatives



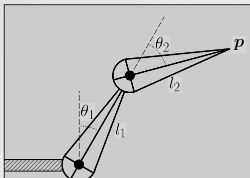
$$\frac{\partial p_z}{\partial \theta_1} = -l_1 \sin(\theta_1) \; - l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial p_x}{\partial \theta_1} = \; l_1 \cos(\theta_1) \; + l_2 \cos(\theta_1 + \theta_2)$$

$$\frac{\partial p_z}{\partial \theta_2} = \; - l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial p_x}{\partial \theta_2} = \; + l_2 \cos(\theta_1 + \theta_2)$$

Inverse Kinematics



Direction in Config. Space

$$\theta_1(\theta_*) = c_1 \theta_*$$
$$\theta_2(\theta_*) = c_2 \theta_*$$

$$\frac{d\theta_1}{d\theta_*} = c_1 \qquad \frac{d\theta_2}{d\theta_*} = c_2$$

$$\frac{dp_z(\theta_1(\theta_*), \theta_2(\theta_*))}{d\theta_*} = \frac{dp_z}{d\theta_1}\frac{d\theta_1}{d\theta_*} + \frac{dp_z}{d\theta_2}\frac{d\theta_2}{d\theta_*}$$

$$\frac{dp_z}{d\theta_*} = \frac{dp_z}{d\theta_1} c_1 + \frac{dp_z}{d\theta_2} c_2$$

Inverse Kinematics

The Jacobian (of *p* w.r.t. θ)

$$J_{ij} = \frac{\partial p_i}{\partial \theta_j}$$

Example for two segment arm

$$J = \begin{bmatrix} \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} \\ \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \end{bmatrix}$$

Inverse Kinematics

The Jacobian (of p w.r.t. θ)

$$J = \begin{bmatrix} \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} \\ \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \end{bmatrix}$$

$$\frac{\partial \boldsymbol{p}}{\partial \theta_*} = J \cdot \begin{bmatrix} \frac{\partial \theta_1}{\partial \theta_*} \\ \frac{\partial \theta_2}{\partial \theta_*} \end{bmatrix} = J \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Inverse Kinematics

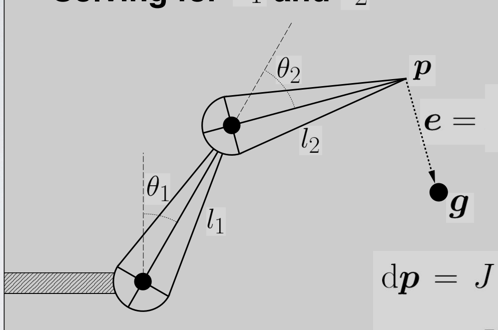
Solving for c_1 and c_2

$$\boldsymbol{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \mathrm{d}\boldsymbol{p} = \begin{bmatrix} \mathrm{d}p_z \\ \mathrm{d}p_x \end{bmatrix}$$

$$\begin{aligned} \mathrm{d}\boldsymbol{p} &= J \cdot \boldsymbol{c} \\ \boldsymbol{c} &= J^{-1} \cdot \mathrm{d}\boldsymbol{p} \end{aligned}$$

Inverse Kinematics

Solving for c_1 and c_2

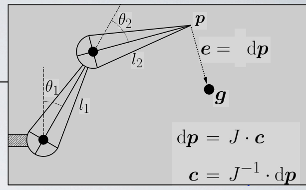


$$\boldsymbol{e} = \mathrm{d}\boldsymbol{p}$$

$$\begin{aligned} \mathrm{d}\boldsymbol{p} &= J \cdot \boldsymbol{c} \\ \boldsymbol{c} &= J^{-1} \cdot \mathrm{d}\boldsymbol{p} \end{aligned}$$

Is the Jacobian invertible?

Inverse Kinematics



Jacobian is not always invertible

- Use pseudo inverse (SVD)

$$J = \begin{bmatrix} \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} \\ \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \end{bmatrix} = J(\theta)$$

Computing a linear approximation

- End effector only locally moves linearly
- So iterate (choosing proper step size) and update Jacobian after each step
- Choosing step size requires line search at each step
  - Choose some step size (say 5 degrees) and compute how to update joint parameters
  - Calculate distance of end effector from goal
  - If distance decreased take step
  - Is distance did not decrease set parameters to be half the current change and try again

Inverse Kinematics

More complex systems

- More complex joints (prism and ball)
- More links
- Hard constraints (joint limits)
- Multiple criteria and multiple chains

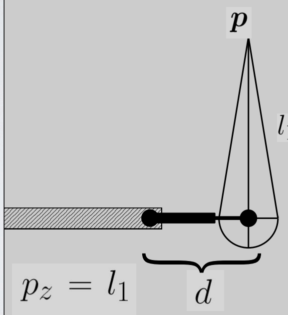
Inverse Kinematics

Some issues

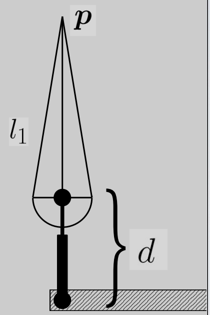
- How to pick from multiple solutions?
- Robustness when no solutions
- Contradictory solutions
- Smooth interpolation
  - Interpolation aware of constraints

Inverse Kinematics

Prism Joints



$$p_z = l_1$$
$$p_x = d$$

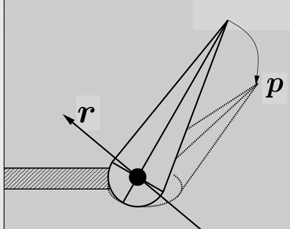


$$p_z = l_1 + d$$
$$p_x = 0$$

Inverse Kinematics

Ball Joints

$$\begin{aligned}\boldsymbol{p} &= \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \boldsymbol{x}) \\ &+ \sin(||\boldsymbol{r}||)(\hat{\boldsymbol{r}} \times \boldsymbol{x}) \\ &- \cos(||\boldsymbol{r}||)(\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{x}))\end{aligned}$$



Inverse Kinematics

Ball Joints (moving axis)

$$\mathrm{d}\boldsymbol{p} = [\mathrm{d}\boldsymbol{r}]\cdot e^{[\boldsymbol{r}]}\cdot\boldsymbol{x} = [\mathrm{d}\boldsymbol{r}]\cdot\boldsymbol{p} = \underbrace{-[\boldsymbol{p}]}_{\text{That is the Jacobian for this joint}}\cdot\mathrm{d}\boldsymbol{r}$$

$$[\boldsymbol{r}] = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$$

$$[\boldsymbol{r}] \cdot \boldsymbol{x} = \boldsymbol{r} \times \boldsymbol{x}$$

Inverse Kinematics

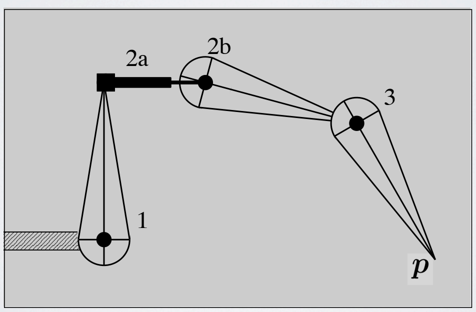
Ball Joints (fixed axis)

$$\mathrm{d}\boldsymbol{p} = (\mathrm{d}\theta)[\hat{\boldsymbol{r}}]\cdot\boldsymbol{x} = \underbrace{-[\boldsymbol{x}]}_{\text{That is the Jacobian for this joint}}\cdot\,\hat{\boldsymbol{r}}\mathrm{d}\theta$$

Inverse Kinematics

Many links / joints

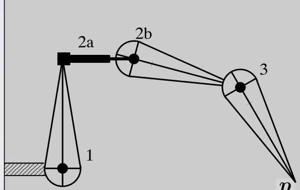
- Need a generic method for building Jacobian



Inverse Kinematics

Can't just concatenate individual matrices

$$\cancel{\tilde{J} = [J_3\ J_{2\mathrm{b}}\ J_{2\mathrm{a}}\ J_{1\mathrm{b}}]}$$



$$\boldsymbol{d} = \begin{bmatrix} d_3 \\ d_{2\mathrm{b}} \\ d_{2\mathrm{a}} \\ d_{1\mathrm{b}} \end{bmatrix}$$

$$\boxed{\mathrm{d}\boldsymbol{p} \neq \tilde{J}\cdot\mathrm{d}\boldsymbol{d}}$$

Inverse Kinematics

Transformation from body to world

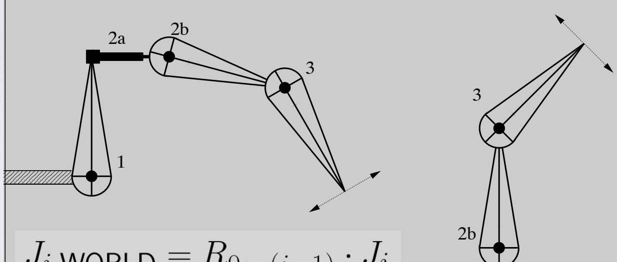
$$X_{0\leftarrow i} = \prod_{j=1}^{i} X_{(j-1)\leftarrow j} = X_{0\leftarrow 1} \cdot X_{1\leftarrow 2} \cdots$$

Rotation from body to world

$$R_{0\leftarrow i} = \prod_{j=1}^{i} R_{(j-1)\leftarrow j} = R_{0\leftarrow 1} \cdot R_{1\leftarrow 2} \cdots$$

Inverse Kinematics

Need to transform Jacobians to common coordinate system (WORLD)



$$J_{i,\mathsf{WORLD}} = R_{0\leftarrow(i-1)} \cdot J_i$$

Inverse Kinematics

$$J = \begin{bmatrix} R_{0\leftarrow 2\mathrm{b}} \cdot & J_3(\theta_3, \boldsymbol{p_3}) \\ R_{0\leftarrow 2\mathrm{a}} \cdot & J_{2\mathrm{b}}(\theta_{2\mathrm{b}}, X_{2\mathrm{b}\leftarrow 3} \cdot \boldsymbol{p_3}) \\ R_{0\leftarrow 1} \cdot & J_{2\mathrm{a}}(\theta_{2\mathrm{a}}, X_{2\mathrm{a}\leftarrow 3} \cdot \boldsymbol{p_3}) \\ & J_1(\theta_1, X_{1\leftarrow 3} \cdot \boldsymbol{p_3}) \end{bmatrix}^{\mathsf{T}}$$

Note: Each row in the above should be transposed....

$$\boldsymbol{d} = \begin{bmatrix} d_3 \\ d_{2\mathrm{b}} \\ d_{2\mathrm{a}} \\ d_{1\mathrm{b}} \end{bmatrix}$$

$$\boxed{\mathrm{d}\boldsymbol{p} = J \cdot \mathrm{d}\boldsymbol{d}}$$

Suggested Reading

Numerical Methods for Inverse Kinematics by Niels Joubert (see wiki)

Advanced Animation and Rendering Techniques by Watt and Watt

- Chapters 15 and 16