CS-184: Computer Graphics

Lecture 5: 3D Transformations & Rotations

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Slides based on those of James O'Brien and Adrien Treuille

Today

Transformations in 3D

Rotations

- Matrices
- Euler angles
- Exponential maps
- Quaternions

3D Transformations

Generally, the extension from 2D to 3D is straightforward

- Vectors get longer by one
- Matrices get extra column and row
- SVD still works the same way
- Scale, Translation, and Shear all basically the same

Rotations get interesting

Translations

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

For 2D

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 3D

Scales

$$\tilde{\mathbf{A}} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For 2D

$$\tilde{\mathbf{A}} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 3D

(Axis-aligned!)

Shears

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & h_{xy} & 0 \\ h_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For 2D

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 3D

(Axis-aligned!)

Shears

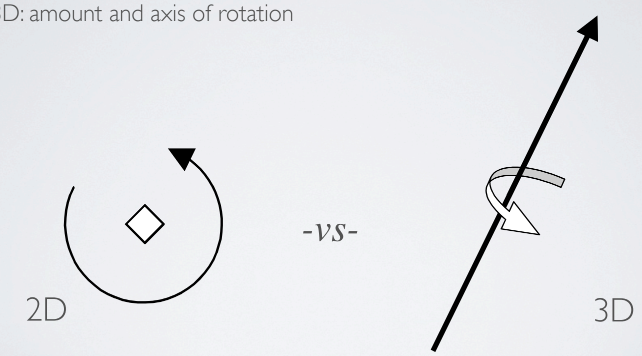
$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shears y into x

Rotations

3D Rotations fundamentally more complex than in 2D

- 2D: amount of rotation
- 3D: amount and axis of rotation



Rotations

Rotations still orthonormal

$$\mathrm{Det}(\mathbf{R}) = 1 \neq -1$$

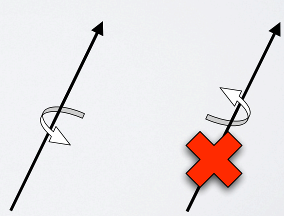
Preserve lengths and distance to origin

3D rotations DO NOT COMMUTE!

Right-hand rule

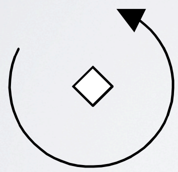
Unique matrices

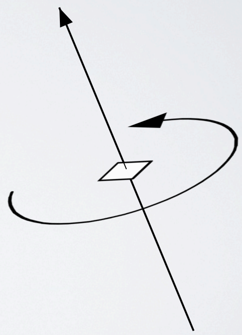
DO NOT COMMUTE!



Axis-aligned 3D Rotations

2D rotations implicitly rotate about a third out of plane axis

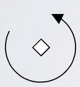




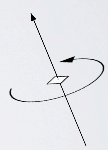
Axis-aligned 3D Rotations

2D rotations implicitly rotate about a third out of plane axis

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Note: looks same as $\tilde{\mathbf{R}}$



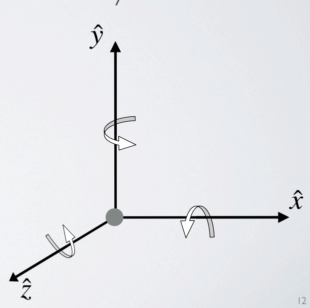
Axis-aligned 3D Rotations

$$\mathbf{R}_{\hat{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\hat{y}} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\hat{z}} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

“Z is in your face”



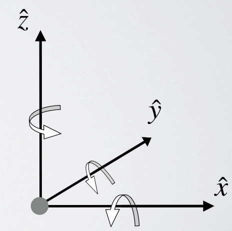
Axis-aligned 3D Rotations

$$\mathbf{R}_{\hat{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\hat{y}} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\hat{z}} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Also right handed "Zup"



Axis-aligned 3D Rotations

Also known as "direction-cosine" matrices

$$\mathbf{R}_{\hat{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \mathbf{R}_{\hat{y}} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

$$\mathbf{R}_{\hat{z}} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Arbitrary Rotations

Can be built from axis-aligned matrices:

$$\mathbf{R} = \mathbf{R}_{\hat{z}} \cdot \mathbf{R}_{\hat{y}} \cdot \mathbf{R}_{\hat{x}}$$

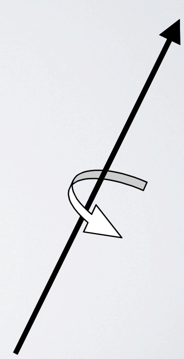
Result due to Euler... hence called

Euler Angles

Easy to store in vector

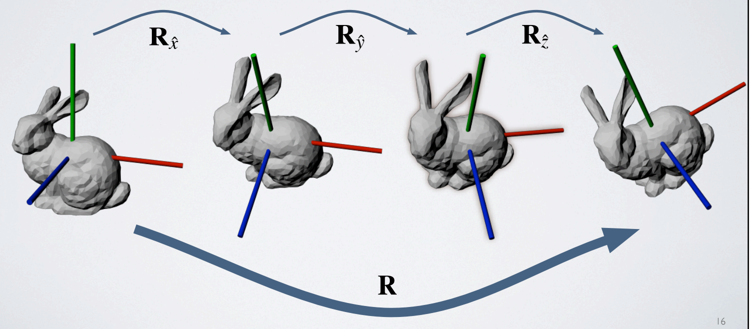
But NOT a vector.

$$\mathbf{R} = \operatorname{rot}(x, y, z)$$



Arbitrary Rotations

$$\mathbf{R} = \mathbf{R}_{\hat{z}} \cdot \mathbf{R}_{\hat{y}} \cdot \mathbf{R}_{\hat{x}}$$



Euler Angles and Gimbal Lock

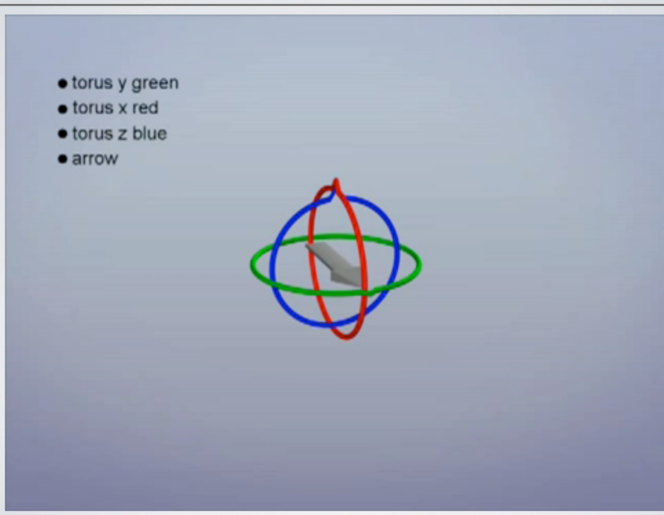
Order of Euler angles matters

Gimbal-lock

Moving -vs- fixed axes

- Reverse of each other

Gimbal Lock



Exponential Maps

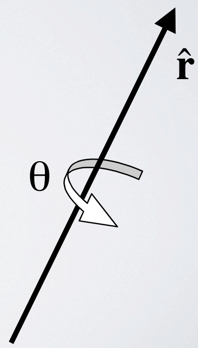
Direct representation of arbitrary rotation

AKA: axis-angle, angular displacement vector

Rotate θ degrees about some axis

Encode θ as length of vector

$$\theta = |\mathbf{r}|$$



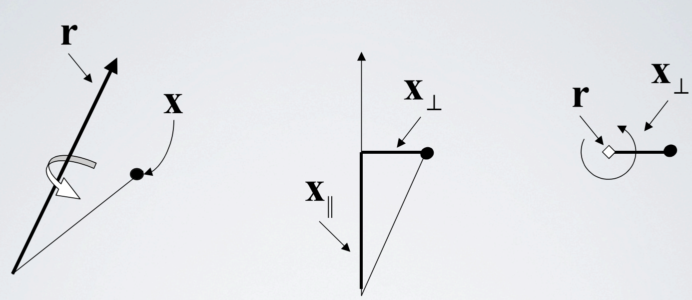
Exponential Maps

Given vector $\mathbf{r}$, how to get matrix $\mathbf{R}$

Method from text:

1. rotate about x axis to put $\mathbf{r}$ into the x-y plane
2. rotate about z axis align $\mathbf{r}$ with the x axis
3. rotate θ degrees about x axis
4. undo #2 and then #1
5. composite together

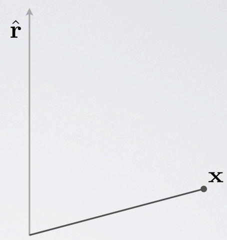
Exponential Maps



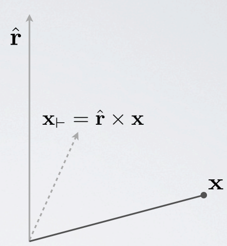
Vector expressing a point has two parts

- $\mathbf{x}_\parallel$ does not change
- $\mathbf{x}_\perp$ rotates like a 2D point

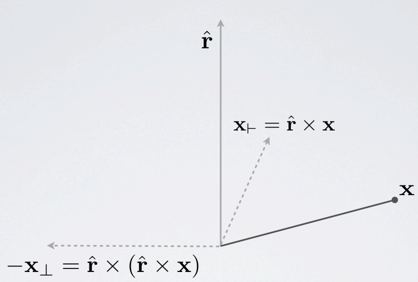
Exponential Maps



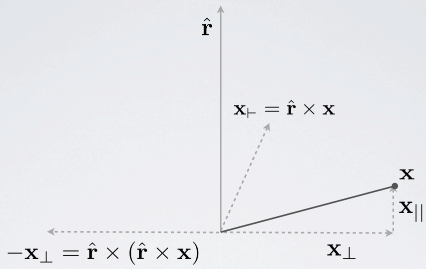
Exponential Maps



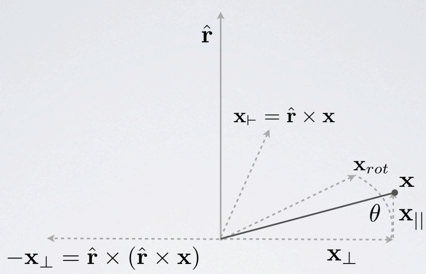
Exponential Maps



Exponential Maps

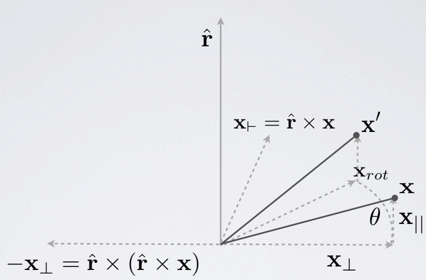


Exponential Maps



$$\mathbf{x}_{rot} = \mathbf{x}_{\vdash} \sin(\theta) + \mathbf{x}_{\perp} \cos(\theta)$$

Exponential Maps



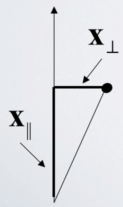
$$\mathbf{x}' = \mathbf{x}_{||} + \mathbf{x}_{rot}$$

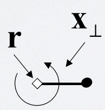
$$\mathbf{x}' = \mathbf{x}_{||} + \mathbf{x}_{\vdash} \sin(\theta) + \mathbf{x}_{\perp} \cos(\theta)$$

Exponential Maps

Rodriguez Formula

$$\mathbf{x}' = \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{x}) + \sin(\theta)(\hat{\mathbf{r}}\times\mathbf{x}) - \cos(\theta)(\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{x}))$$





Linear in $\mathbf{x}$

Actually a minor variation ...

Exponential Maps

Building a matrix form:

$$\mathbf{x}' = ((\hat{\mathbf{r}}\hat{\mathbf{r}}^t) + \sin(\theta)(\hat{\mathbf{r}}\times) - \cos(\theta)(\hat{\mathbf{r}}\times)(\hat{\mathbf{r}}\times))\,\mathbf{x}$$

$$(\hat{\mathbf{r}}\times) = \begin{bmatrix} 0 & -\hat{r}_z & \hat{r}_y \\ \hat{r}_z & 0 & -\hat{r}_x \\ -\hat{r}_y & \hat{r}_x & 0 \end{bmatrix}$$

Antisymmetric matrix

$$(\mathbf{a}\times)\mathbf{b} = \mathbf{a}\times\mathbf{b}$$

Easy to verify by expansion

Exponential Maps

- Allows tumbling
- No gimbal-lock!
- Orientations are space within π-radius ball
- Nearly unique representation
- Singularities on shells at 2π
- Nice for interpolation

Exponential Maps

Why exponential?

Recall series expansion of e^x

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

31

Exponential Maps

Why exponential?

Recall series expansion of e^x

Euler: what happens if you put in $i\theta$ for x

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{-\theta^2}{2!} + \frac{-i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= \left(1 + \frac{-\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\frac{\theta}{1!} + \frac{-\theta^3}{3!} + \cdots\right)$$

$$= \cos(\theta) + i\sin(\theta)$$

32

Exponential Maps

Why exponential?

$$e^{(\hat{\mathbf{r}}\times)\theta} = \mathbf{I} + \frac{(\hat{\mathbf{r}}\times)\theta}{1!} + \frac{(\hat{\mathbf{r}}\times)^2\theta^2}{2!} + \frac{(\hat{\mathbf{r}}\times)^3\theta^3}{3!} + \frac{(\hat{\mathbf{r}}\times)^4\theta^4}{4!} + \cdots$$

But notice that: $(\hat{\mathbf{r}}\times)^3 = -(\hat{\mathbf{r}}\times)$

$$e^{(\hat{\mathbf{r}}\times)\theta} = \mathbf{I} + \frac{(\hat{\mathbf{r}}\times)\theta}{1!} + \frac{(\hat{\mathbf{r}}\times)^2\theta^2}{2!} + \frac{-(\hat{\mathbf{r}}\times)\theta^3}{3!} + \frac{-(\hat{\mathbf{r}}\times)^2\theta^4}{4!} + \cdots$$

33

Exponential Maps

$$e^{(\hat{\mathbf{r}}\times)\theta} = \mathbf{I} + \frac{(\hat{\mathbf{r}}\times)\theta}{1!} + \frac{(\hat{\mathbf{r}}\times)^2\theta^2}{2!} + \frac{-(\hat{\mathbf{r}}\times)\theta^3}{3!} + \frac{-(\hat{\mathbf{r}}\times)^2\theta^4}{4!} + \cdots$$

$$e^{(\hat{\mathbf{r}}\times)\theta} = (\hat{\mathbf{r}}\times)\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots\right) + \mathbf{I} + (\hat{\mathbf{r}}\times)^2\left(+\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)$$

$$e^{(\hat{\mathbf{r}}\times)\theta} = (\hat{\mathbf{r}}\times)\sin(\theta) + \mathbf{I} + (\hat{\mathbf{r}}\times)^2(1-\cos(\theta))$$

But notice that: $\mathbf{I} + (\hat{\mathbf{r}}\times)^2 = \hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{t}}$

$$e^{(\hat{\mathbf{r}}\times)\theta} = \hat{\mathbf{r}}\hat{\mathbf{r}}^{\mathrm{t}} + (\hat{\mathbf{r}}\times)\sin(\theta) - (\hat{\mathbf{r}}\times)^2\cos(\theta)$$

Quaternions

More popular than exponential maps

Natural extension of $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

Due to Hamilton (1843)

- Interesting history
- Involves "hermaphroditic monsters"

Quaternions

Uber-Complex Numbers

$$\mathsf{q} = (z_1, z_2, z_3, s) = (\mathbf{z}, s)$$

$$\mathsf{q} = iz_1 + jz_2 + kz_3 + s$$

$$i^2 = j^2 = k^2 = -1$$

$$\begin{array}{ll} ij = k & ji = -k \\ jk = i & kj = -i \\ ki = j & ik = -j \end{array}$$

Quaternions

Multiplication natural consequence of defn.

$$\mathsf{q}\cdot\mathsf{p} = \left(\mathbf{z}_q s_p + \mathbf{z}_p s_q + \mathbf{z}_p \times \mathbf{z}_q \ ,\ \ s_p s_q - \mathbf{z}_p \cdot \mathbf{z}_q\right)$$

Conjugate

$$\mathsf{q}^* = (-\mathbf{z}, s)$$

Magnitude

$$||\mathsf{q}||^2 = \mathbf{z}\cdot\mathbf{z} + s^2 = \mathsf{q}\cdot\mathsf{q}^*$$

Quaternions

Vectors as quaternions

$$\mathsf{v} = (\mathbf{v}, 0)$$

Rotations as quaternions

$$\mathsf{r} = (\hat{\mathbf{r}}\sin\frac{\theta}{2}, \cos\frac{\theta}{2})$$

Rotating a vector

$$\mathsf{x}' = \mathsf{r}\cdot\mathsf{x}\cdot\mathsf{r}^*$$

Composing rotations

$$\mathsf{r} = \mathsf{r}_1\cdot\mathsf{r}_2$$

Compare to Exp. Map

Quaternions

No tumbling

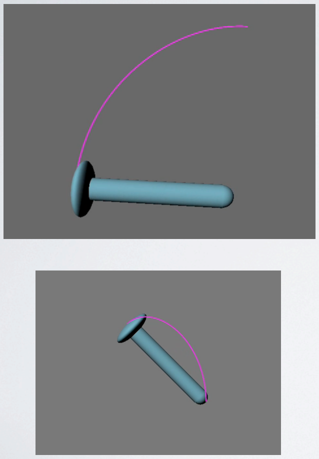
No gimbal-lock

Orientations are "double unique"

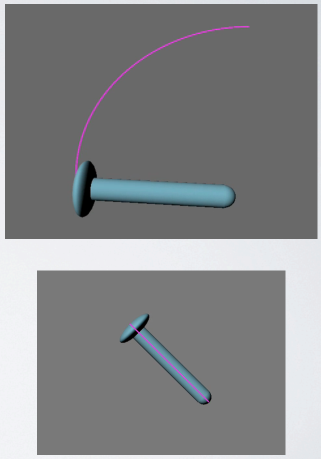
Surface of a 3-sphere in 4D $\quad ||\mathsf{r}|| = 1$

Nice for interpolation

Interpolation



Euler Angles



Quaternions

Rotation Matrices

Consider:

$$\mathbf{RI} = \begin{bmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Columns are coordinate axes after transformation (true for general matrices)

Rows represent axes that will rotate into canonical xyz-axes after rotation (not true for general matrices)

Rotation Matrices

Eigen system

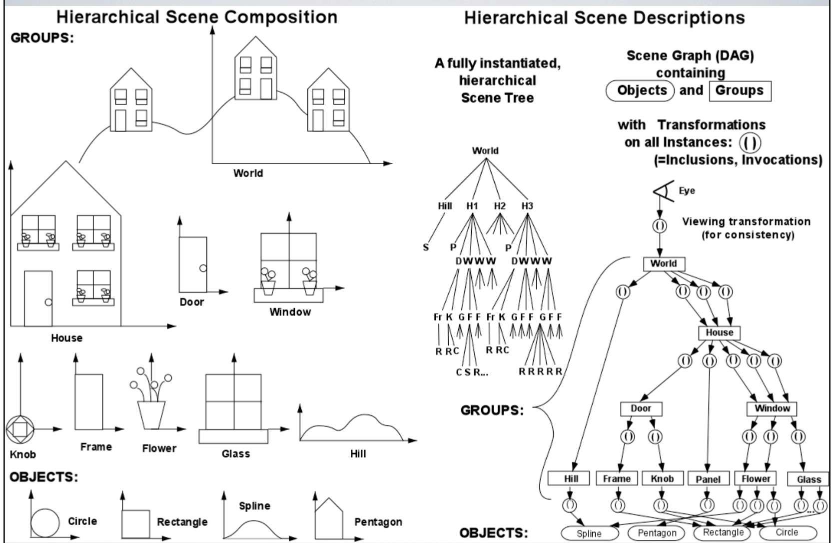
- One real eigenvalue
- Real axis is axis of rotation
- Imaginary values are 2D rotation as complex number

Logarithmic formula

$$(\hat{\mathbf{r}}\times) = \ln(\mathbf{R}) = \frac{\theta}{2\sin\theta}(\mathbf{R} - \mathbf{R}^{\mathsf{T}})$$

$$\theta = \cos^{-1}\left(\frac{\mathrm{Tr}(\mathbf{R}) - 1}{2}\right)$$

Similar formulae as for exponential...



Scene Graphs

Draw scene using both pre-and-post-order traversal

- Apply node, draw children, undo node (if applicable)

Nodes can do pretty much anything

- Geometry, transformations, groups, color, switch, scripts, *etc.*
- Node types are application/implementation specific

Requires a stack to implement "undo" node post children

Nodes can cache their children

Instances make it a DAG, not strictly a tree

Will use these trees later for bounding box trees

Note:

Rotation stuff in the book is a bit weak... luckily you have these nice slides!