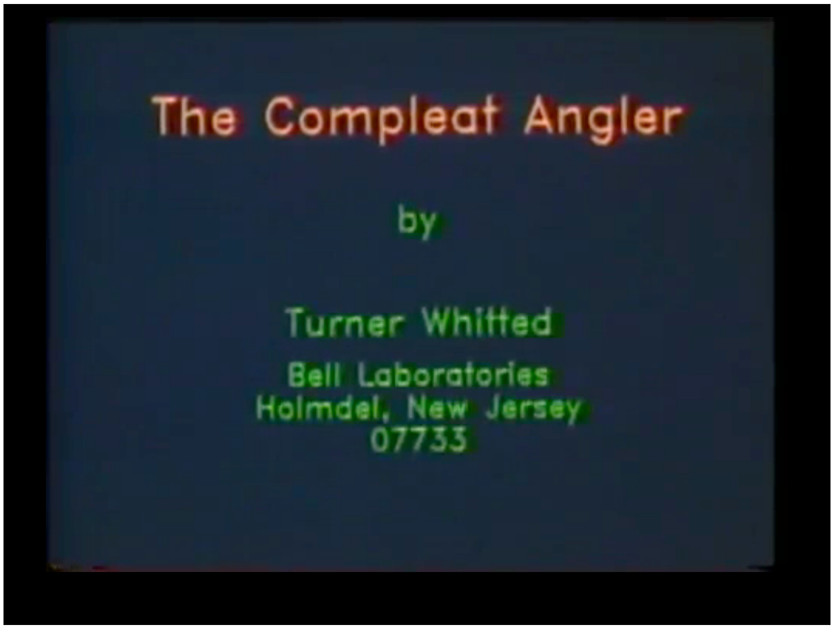


CS-184: Computer Graphics

Lecture 4: 2D Transformations

Maneesh Agrawala
University of California, Berkeley

Slides based on those of James O'Brien and Adrien Treuille

Announcements

~~Assignment 2: due Fri Sep 10 by 11pm~~

Assignment 3: due Fri Sep 17 by 11pm

Today

2D Transformations

- "Primitive" Operations
  - Scale, Rotate, Shear, Flip, Translate
- Homogenous Coordinates
- SVD
- Start thinking about rotations...

Introduction

Transformation:

An operation that changes one configuration into another

For images, shapes, ***etc.***

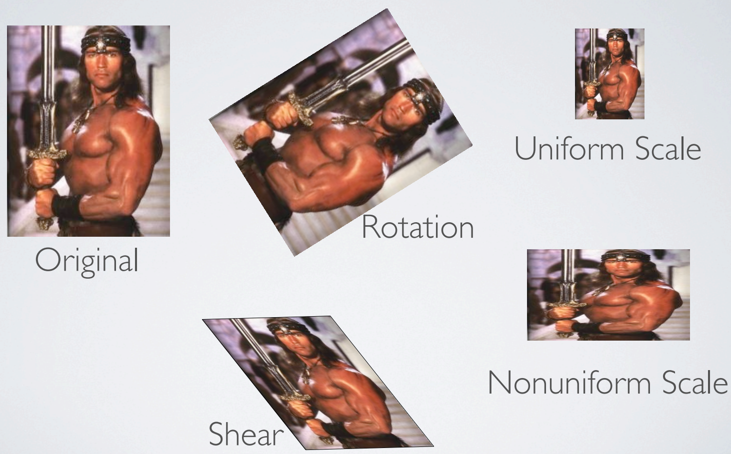
Geometric transformation maps positions that define object to new positions

Linear transformation means the transformation is defined by a linear function... which is what matrices are good for.

Some Examples

Original

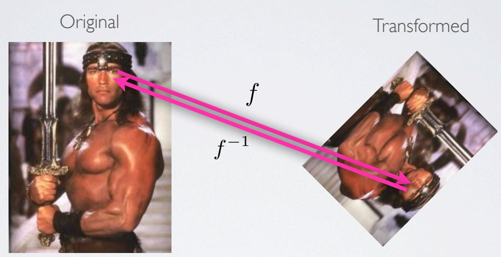
Some Examples



Images from *Conan The Destroyer*, 1984

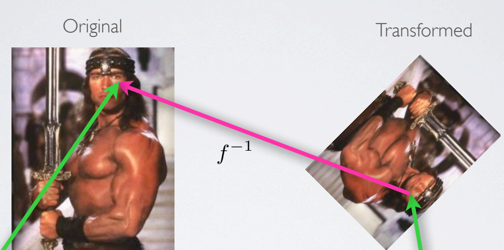
Mapping Function

$f(\mathbf{p}) = \mathbf{p}'$ Maps points in original image $\mathbf{p} = (\mathbf{x}, \mathbf{y})$
to point in transformed image $\mathbf{p}' = (\mathbf{x}', \mathbf{y}')$



Mapping Function

$f(\mathbf{p}) = \mathbf{p}'$ Maps points in original image $\mathbf{p} = (\mathbf{x}, \mathbf{y})$
to point in transformed image $\mathbf{p}' = (\mathbf{x}', \mathbf{y}')$



$c(\mathbf{p}) = [\mathbf{195}, \mathbf{120}, \mathbf{58}]$

$c(\mathbf{p}')$
$= c(f^{-1}(\mathbf{p}')) = c(\mathbf{p})$

Linear -vs- Nonlinear

Nonlinear (swirl)

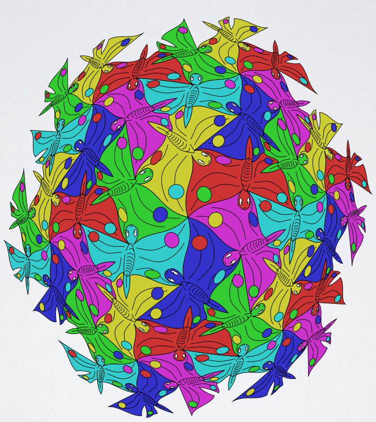
Linear (shear)

Geometric -vs- Color Space

Color Space Transform
(edge finding)

Linear Geometric
(flip)

Instancing



M.C. Escher, from Ghostscript 8.0 Distribution

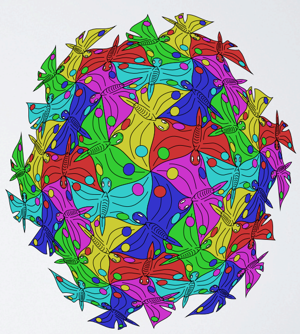
Instancing

Carlo Sequin

Instancing

Reuse geometric descriptions

Saves memory



Linear is Linear

Composing two linear function is still linear

Transform polygon by transforming vertices

$$f(x) = a + bx \qquad g(f) = c + df$$

$$g(x) = c + df(x) = c + ad + bdx$$

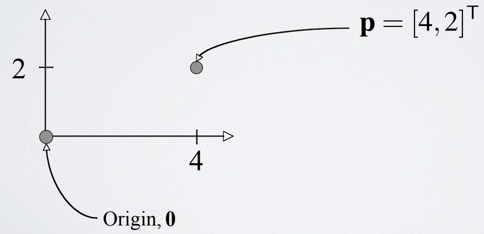
$$g(x) = a' + b'x$$

Points in Space

Represent point in space by vector in R^n

- Relative to some origin!
- Relative to some coordinate axes!

Later we'll add something extra...



Linear Functions in 2D

$$x' = f(x,y) = c_1 + c_2 x + c_3 y$$
$$y' = f(x,y) = d_1 + d_2 x + d_3 y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{t} + \mathbf{M} \cdot \mathbf{x}$$

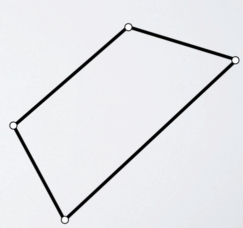
Linear is Linear

Polygons defined by points

Edges defined by interpolation between two points

Interior defined by interpolation between all points

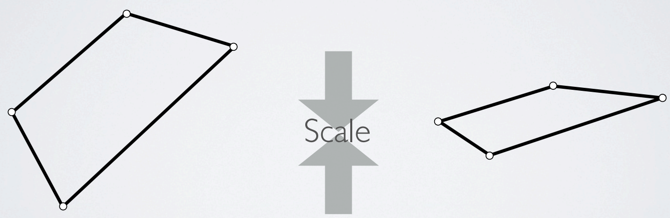
Linear **interpolation**



Linear is Linear

Composing two linear function is still linear

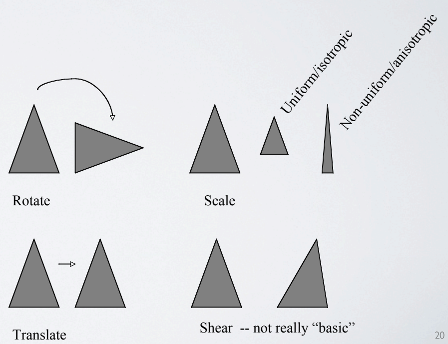
Transform polygon by transforming vertices



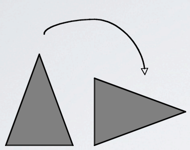
Basic Transformations

Basic transforms are: rotate, scale, and translate

Shear is a composite transformation!

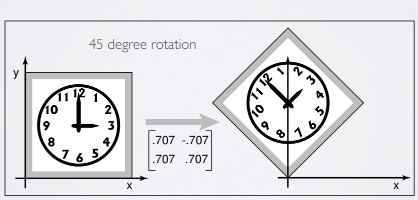


Rotations



Rotate

$$\mathbf{p}' = \begin{bmatrix} \mathrm{Cos}(\theta) & -Sin(\theta) \\ Sin(\theta) & \mathrm{Cos}(\theta) \end{bmatrix} \mathbf{p}$$



Rotations

Rotations are positive counter-clockwise

Consistent w/ right-hand rule

Don't be different...

Note:

- rotate by zero degrees give identity
- rotations are modulo 360 (or 2π)

Rotations

Preserve lengths and distance to origin

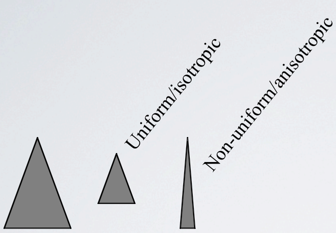
Rotation matrices are orthonormal

$$\mathrm{Det}(\mathbf{R}) = 1 \neq -1$$

In 2D rotations commute...

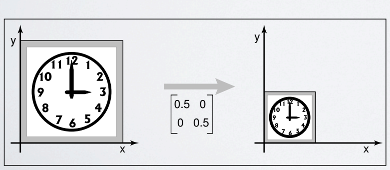
- But in 3D they won't!

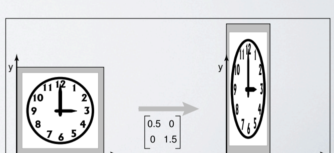
Scales



Scale

$$\mathbf{p}' = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \mathbf{p}$$

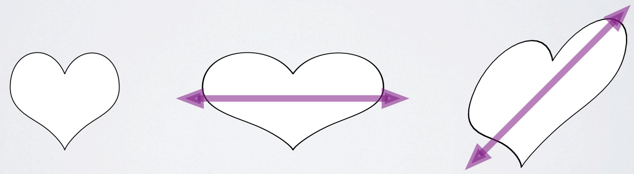




Scales

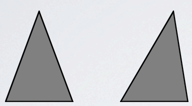
Diagonal matrices

- Diagonal parts are scale in X and scale in Y directions
- Negative values flip
- Two negatives make a positive (180 deg. rotation)
- Really, axis-aligned scales



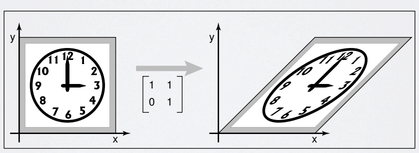
Not axis-aligned...

Shears



Shear

$$\mathbf{p}' = \begin{bmatrix} 1 & H_{yx} \\ H_{xy} & 1 \end{bmatrix} \mathbf{p}$$



Shears

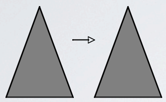
Shears are not really primitive transforms

Related to non-axis-aligned scales

More shortly.....

Translation

This is the not-so-useful way:



Translate

$$\mathbf{p}' = \mathbf{p} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

Note that its not like the others.

Arbitrary Matrices

For everything but translations we have:

$$\mathbf{x}' = \mathbf{A} \cdot \mathbf{x}$$

Soon, translations will be assimilated as well

What does an arbitrary matrix mean?

Singular Value Decomposition

For any matrix, $\mathbf{A}$, we can write SVD:

$$\mathbf{A} = \mathbf{QSR}^\mathsf{T}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are orthonormal and $\mathbf{S}$ is diagonal

Can also write Polar Decomposition

$$\mathbf{A} = \mathbf{QRSR}^\mathsf{T}$$

where $\mathbf{Q}$ is still orthonormal

not the same $\mathbf{Q}$

Decomposing Matrices

We can force $\mathbf{Q}$ and $\mathbf{R}$ to have $\mathrm{Det}=1$ so they are rotations

Any matrix is now:

- Rotation:Rotation:Scale:Rotation
- See, shear is just a mix of rotations and scales

Composition

Matrix multiplication composites matrices

$$\mathbf{p}' = \mathbf{BAp}$$

"Apply $\mathbf{A}$ to $\mathbf{p}$ and then apply $\mathbf{B}$ to that result."

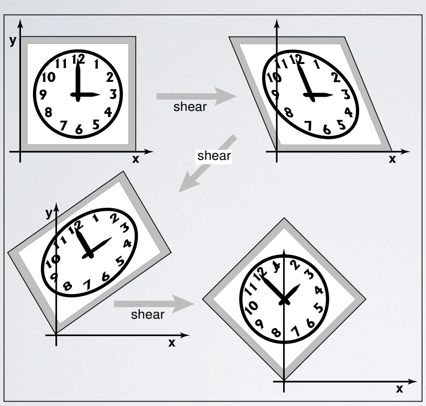
$$\mathbf{p}' = \mathbf{B}(\mathbf{Ap}) = (\mathbf{BA})\mathbf{p} = \mathbf{Cp}$$

Several transformations composited to one

Translations still left out...

$$\mathbf{p}' = \mathbf{B}(\mathbf{Ap} + \mathbf{t}) = \cancel{\mathbf{BA}}\mathbf{p} + \mathbf{Bt} = \mathbf{Cp} + \mathbf{u}$$

Composition



Transformations built up from others

SVD builds from scale and rotations

Also build other ways

i.e. 45 deg rotation built from shears

Homogeneous Coordinates

Move to one higher dimensional space

- Append a 1 at the end of the vectors

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \qquad \tilde{\mathbf{p}} = \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

- For *directions* the extra coordinate is a zero

Homogeneous Translation

$$\tilde{\mathbf{p}}' = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{p}}' = \tilde{\mathbf{A}}\tilde{\mathbf{p}}$$

The tildes are for clarity to distinguish homogenized from non-homogenized vectors.

Homogeneous Others

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now everything looks the same...
Hence the term "homogenized!"

Compositing Matrices

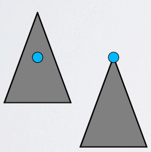
Rotations and scales always about the origin

How to rotate/scale about another point?

Rotate About Arb. Point

Step 1: Translate point to origin

Translate (-**C**)



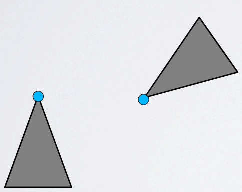
Rotate About Arb. Point

Step 1: Translate point to origin

Step 2: Rotate as desired

Translate (-**C**)

Rotate (θ)

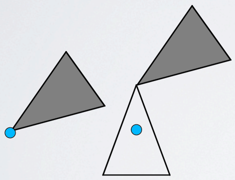


Rotate About Arb. Point

Step 1: Translate point to origin

Step 2: Rotate as desired

Step 3: Put back where it was



Translate (-**C**)

Rotate (θ)

Translate (**C**)

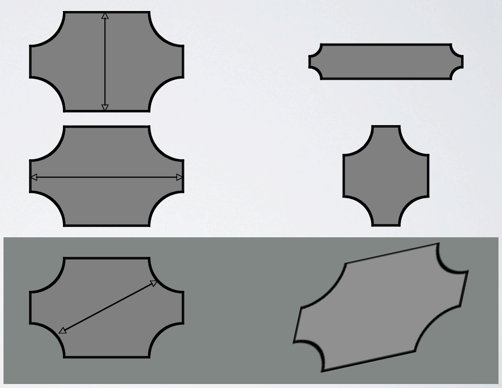
$$\tilde{\mathbf{p}}' = (-\mathbf{T})\mathbf{R}\mathbf{T}\tilde{\mathbf{p}} = \mathbf{A}\tilde{\mathbf{p}}$$

Don't negate the 1...

Scale About Arb. Axis

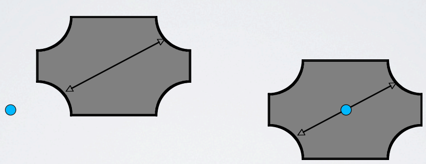
Diagonal matrices scale about coordinate axes only:

Not axis-aligned



Scale About Arb. Axis

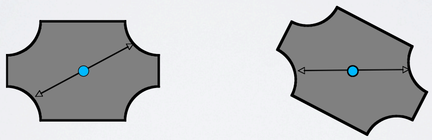
Step 1: Translate axis to origin



Scale About Arb. Axis

Step 1: Translate axis to origin

Step 2: Rotate axis to align with one of the coordinate axes

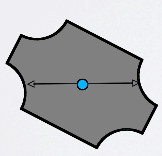


Scale About Arb. Axis

Step 1: Translate axis to origin

Step 2: Rotate axis to align with one of the coordinate axes

Step 3: Scale as desired



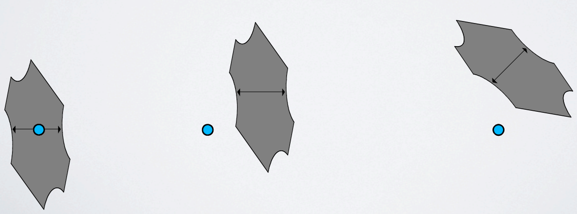
Scale About Arb. Axis

Step 1: Translate axis to origin

Step 2: Rotate axis to align with one of the coordinate axes

Step 3: Scale as desired

Steps 4&5: Undo 2 and 1 (reverse order)



Order Matters!

The order that matrices appear in matters

$$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{BA}$$

Some special cases work, but they are special

But matrices are associative

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})$$

Think about efficiency when you have many points to transform...

Matrix Inverses

In general: $\mathbf{A}^{-1}$ undoes effect of $\mathbf{A}$

Special cases:

- Translation: negate t_x and t_y
- Rotation: transpose
- Scale: invert diagonal (axis-aligned scales)

Others:

- Invert matrix
- Invert SVD matrices

Point Vectors / Direction

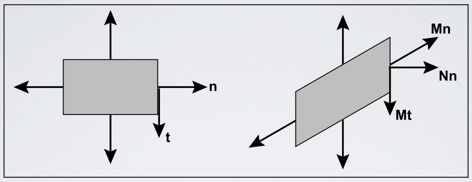
Points in space have a 1 for the "w" coordinate

What should we have for $\mathbf{a} - \mathbf{b}$?

- $w = 0$
- Directions not the same as positions
- Difference of positions is a direction
- Position + direction is a position
- Direction + direction is a direction
- Position + position is nonsense

Some Things Require Care

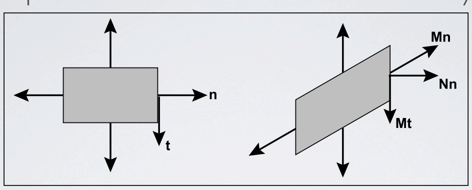
For example normals transform abnormally



$\mathbf{n^T t} = 0 \quad \mathbf{t_M} = \mathbf{Mt}$ find $\mathbf{N}$ such that $\mathbf{n_N^T t_M} = 0$

Some Things Require Care

For example normals transform abnormally

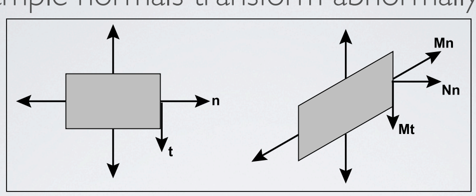


$\mathbf{n^T t} = 0 \quad \mathbf{t_M} = \mathbf{Mt}$ find $\mathbf{N}$ such that $\mathbf{n_N^T t_M} = 0$

$$\mathbf{n^T t} = \mathbf{n^T I t} = \mathbf{n^T M^{-1} M t} = 0$$

Some Things Require Care

For example normals transform abnormally



$\mathbf{n^T t} = 0 \quad \mathbf{t_M} = \mathbf{Mt}$ find $\mathbf{N}$ such that $\mathbf{n_N^T t_M} = 0$

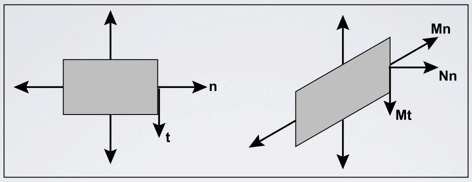
$$\mathbf{n^T t} = \mathbf{n^T I t} = \mathbf{n^T M^{-1} M t} = 0$$

$$(\mathbf{n^T M^{-1}})\mathbf{t_M} = 0$$

$$\mathbf{n_N^T} = \mathbf{n^T M^{-1}}$$

Some Things Require Care

For example normals transform abnormally



$\mathbf{n^T t} = 0 \quad \mathbf{t_M} = \mathbf{Mt}$ find $\mathbf{N}$ such that $\mathbf{n_N^T t_M} = 0$

$$\mathbf{n^T t} = \mathbf{n^T I t} = \mathbf{n^T M^{-1} M t} = 0$$
$$(\mathbf{n^T M^{-1}})\mathbf{t_M} = 0$$
$$\mathbf{n_N^T} = \mathbf{n^T M^{-1}}$$
$$\mathbf{n_N} = (\mathbf{n^T M^{-1}})^\mathbf{T}$$
$$\mathbf{N} = (\mathbf{M^{-1}})^\mathbf{T}$$

See book for details

Suggested Reading

Fundamentals of Computer Graphics by Pete Shirley

- Chapter 6
- And re-read chapter 5 if your linear algebra is rusty!